

# A mean value formula of sub- $p$ -Laplace parabolic equations on the Heisenberg group \*

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**Abstract.** We derive two equivalent definitions of the viscosity solutions to the homogeneous sub- $p$ -Laplace parabolic equations on the Heisenberg group, and characterize the viscosity solutions in terms of an asymptotic mean value formula, when  $1 < p \leq \infty$ . Moreover, we construct an example to show that these formulae do not hold in non-asymptotic sense.

**Key Words** Heisenberg group, sub- $p$ -Laplace parabolic equation, viscosity solution

**Mathematic Subject Classification.** 35D40, 35K92, 35R03.

## 1 Introduction

Mean value properties for solutions to elliptic and parabolic partial differential equations are important tools for the study of their properties. It is well known that a basic property of harmonic functions is the mean value property [21]. More precisely,  $u$  is a harmonic function in a domain  $\Omega \subset \mathbb{R}^n$  (that is  $u$  satisfies  $\Delta u = 0$  in  $\Omega$ ) if and only if  $u$  satisfies the mean value formula

$$u(x) = \oint_{B_\varepsilon(x)} u(y) dy$$

whenever  $B_\varepsilon(x) \subset \Omega$  and  $\oint_E f$  denotes the average of  $f$  over the set  $E$ . In addition, an asymptotic mean value formula holds for some nonlinear cases as well. Manfredi et al. [27] characterized  $p$ -harmonic functions by means of asymptotic mean value properties that hold in the so called viscosity sense (see Definition 1.2 below). More precisely, they proved that the asymptotic mean value formula

$$u(x) = \frac{\alpha}{2} \left( \max_{\bar{B}_\varepsilon(x)} u + \min_{\bar{B}_\varepsilon(x)} u \right) + \beta \oint_{B_\varepsilon(x)} u(y) dy + o(\varepsilon^2) \quad \text{as } \varepsilon \rightarrow 0$$

holds in the viscosity sense for all  $x \in \Omega$  if and only if  $u$  is a viscosity solution of

$$-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0,$$

where the constants  $\alpha$  and  $\beta$  are given by

$$\alpha = \frac{p-2}{p+n} \quad \text{and} \quad \beta = \frac{2+n}{p+n}.$$

The mean value properties of  $p$ -Laplace parabolic equation were proved by Manfredi et al. [28]. In fact, they proved that the asymptotic mean value formula

$$u(t, x) = \frac{\alpha}{2} \oint_{t-\varepsilon^2}^t \left( \max_{y \in \bar{B}_\varepsilon(x)} u(s, y) + \min_{y \in \bar{B}_\varepsilon(x)} u(s, y) \right) ds + \beta \oint_{t-\varepsilon^2}^t \oint_{B_\varepsilon(x)} u(s, y) dy ds + o(\varepsilon^2), \text{ as } \varepsilon \rightarrow 0$$

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holds for every  $(t, x) \in \Omega_T = (0, T) \times \Omega$  in the viscosity sense if and only if  $u$  is a viscosity solution of

$$(n + p)u_t(t, x) = |\nabla u|^{2-p} \Delta_p u(t, x).$$

The constants  $\alpha$  and  $\beta$  are the same as before.

The purpose of this paper is to extend this result to parabolic equations on the Heisenberg group  $\mathbb{H}^n$ . We recall that  $\mathbb{H}^n$  is the Lie group  $(\mathbb{R}^{2n+1}, \circ)$  equipped with the group action

$$x^0 \circ x = \left( x_1 + x_1^0, \dots, x_{2n} + x_{2n}^0, x_{2n+1} + x_{2n+1}^0 + 2 \sum_{i=1}^n (x_i x_{n+i}^0 - x_i^0 x_{n+i}) \right), \quad (1.1)$$

for  $x = (x_1, \dots, x_n, x_{n+1}, \dots, x_{2n}, x_{2n+1}) = (\bar{x}, x_{2n+1}) \in \mathbb{R}^{2n+1}$ . It is easy to check that (1.1) does indeed make  $\mathbb{R}^{2n} \times \mathbb{R}$  into a group whose identity is the origin, and where the inverse is given by  $x^{-1} = -x$ . Let us denote by  $\delta_\lambda$  the Heisenberg group dilation

$$\delta_\lambda(x_1, \dots, x_{2n}, x_{2n+1}) = (\lambda x_1, \dots, \lambda x_{2n}, \lambda^2 x_{2n+1}), \quad \lambda > 0. \quad (1.2)$$

Then  $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \circ, \delta_\lambda)$  is a homogeneous group. We denote  $Q = 2n + 2$  and call it the homogeneous dimension of  $\mathbb{H}^n$ . For more information on the Heisenberg group, we refer the reader to the monograph [6].

A basis of the Lie algebra of  $\mathbb{H}^n$  is given by

$$\begin{cases} X_i = \frac{\partial}{\partial x_i} + 2x_{n+i} \frac{\partial}{\partial x_{2n+1}}, & i = 1, \dots, n, \\ X_{n+i} = \frac{\partial}{\partial x_{n+i}} - 2x_i \frac{\partial}{\partial x_{2n+1}}, & i = 1, \dots, n, \\ T = \frac{\partial}{\partial x_{2n+1}}. \end{cases} \quad (1.3)$$

From (1.3), it is easy to check that  $X_i$  and  $X_{n+j}$  satisfy

$$[X_i, X_{n+j}] = -4T\delta_{ij}, \quad [X_i, X_j] = [X_{n+i}, X_{n+j}] = 0, \quad i, j = 1, \dots, n.$$

Therefore, the vector fields  $X_i, X_{n+i}$  ( $i = 1, \dots, n$ ) and their first order commutators span the whole Lie Algebra.

Given a function  $u : \mathbb{H}^n \rightarrow \mathbb{R}$ , we consider the full gradient of  $u$

$$\nabla u = (X_1 u, \dots, X_{2n} u, T u)$$

and the horizontal gradient of  $u$

$$\nabla_0 u = (X_1 u, \dots, X_{2n} u)$$

and the symmetrized second horizontal derivative matrix  $(X^2 u)^*$

$$(X^2 u)^* = \frac{1}{2} (X_i X_j u + X_j X_i u).$$

For  $x \in \mathbb{H}^n$ , we define the quasi-distance from the origin

$$\rho(x) = \left( \left( \sum_{i=1}^n (x_i^2 + x_{n+i}^2) \right)^2 + x_{2n+1}^2 \right)^{\frac{1}{4}} \equiv \left( |\bar{x}|^4 + x_{2n+1}^2 \right)^{\frac{1}{4}},$$

which satisfies  $\rho(\delta_\lambda(x)) = \lambda \rho(x)$  and means that  $\rho$  is homogeneous of degree one with respect to the dilation  $\delta_\lambda$ . The associated distance between  $x$  and  $x^0$  is defined by

$$\rho(x; x^0) = \rho((x^0)^{-1} \circ x).$$

In the sequel we let

$$B_r = \{x \in \mathbb{H}^n | \rho(x) < r\}, \quad \partial B_r = \{x \in \mathbb{H}^n | \rho(x) = r\}$$

and call these sets a Heisenberg-ball and a sphere centered at the origin with radius  $r$  respectively. Balls and spheres centered at  $x^0$  are defined by left-translation, i.e.

$$B_r(x^0) = \{x \in \mathbb{H}^n | \rho(x; x^0) < r\}, \quad \partial B_r(x^0) = \{x \in \mathbb{H}^n | \rho(x; x^0) = r\}.$$

Introducing the function

$$\psi(x) = |\nabla_0 \rho|^2 = \frac{|\bar{x}|^2}{\rho(x)^2}, \quad (1.4)$$

we define

$$|B_r| = \int_{B_r} \psi dx \quad \text{and} \quad |\partial B_r| = \frac{d}{dr} |B_r|. \quad (1.5)$$

Gaveau [18] proved the following mean value formula for the sub-Laplace equation

$$\Delta_H u = \sum_{i=1}^{2n} X_i X_i u = 0 \quad (1.6)$$

on  $\mathbb{H}^n$ : let  $u$  solve the equation  $\Delta_H u(x) = 0$ , then

$$u(x^0) = \int_{B_r(x^0)} \psi \left( (x^0)^{-1} \circ x \right) u(x) dx, \quad (1.7)$$

where

$$\psi \left( (x^0)^{-1} \circ x \right) = \frac{|\bar{x} - \bar{(x^0)}|^2}{\rho \left( (x^0)^{-1} \circ (x) \right)^2}.$$

Recently in [25], we characterized sub- $p$ -harmonic functions on  $\mathbb{H}^n$  by asymptotic mean value formulae in the viscosity sense. More precisely, we proved that the asymptotic mean

$$u(x^0) = \frac{\alpha}{2} \left( \max_{\bar{B}_\varepsilon(x^0)} u + \min_{\bar{B}_\varepsilon(x^0)} u \right) + \beta \int_{B_\varepsilon(x^0)} \psi \left( (x^0)^{-1} \circ x \right) u(x) dx + o(\varepsilon^2)$$

holds as  $\varepsilon \rightarrow 0$  for all  $x^0 \in \Omega$  in the viscosity sense if and only if  $u$  is a viscosity solution of

$$-\Delta_H^p u(x) = \sum_{i=1}^{2n} X_i \left( |\nabla_0 u|^{p-2} X_i u \right) (x) = 0. \quad (1.8)$$

In this paper, we study the parabolic version of the sub- $p$ -Laplace equation on  $\mathbb{H}^n$ :

$$u_t(t, x) = |\nabla_0 u|^{2-p} \Delta_H^p u(t, x).$$

Recall that for  $1 < p < \infty$ , we have

$$u_t(t, x) = |\nabla_0 u|^{2-p} \Delta_H^p u = (p-2) \Delta_H^\infty u + \Delta_H u, \quad (1.9)$$

where

$$\Delta_H^\infty u = |\nabla_0 u|^{-2} \left\langle (X^2 u)^* \nabla_0 u, \nabla_0 u \right\rangle = |\nabla_0 u|^{-2} \sum_{i,j=1}^{2n} X_i X_j u \cdot X_i u \cdot X_j u \quad (1.10)$$

denotes the 1-homogeneous version of sub-infinity Laplace equation on  $\mathbb{H}^n$ .

Before proceeding, we would like to mention some motivations related to our research. Since Hörmanders work [22] the study of partial differential equations of sub-elliptic type like (1.6), (1.8)

and (1.10) has received a strong impulse, see, e.g., [3], [4], [7], [11], [12], [13], [17], [31], [32] etc. These equations arise in many different settings: geometric theory of several complex variables, curvature problems for CR-manifolds, sub-Riemannian geometry, diffusion processes, control theory, human vision; see, e.g., [9], [20]. The parabolic counterpart of the operator is also of great relevance; see, e.g., [1], [5], [23], [29].

Let  $T > 0$  and  $\Omega \subset \mathbb{H}^n$  be an open set, and let  $\Omega_T = (0, T) \times \Omega$  be a space-time cylinder. Our main results are the following theorems corresponding to  $p = 2$ ,  $p = \infty$  and  $1 < p < \infty$ , respectively.

**Theorem 1.1.** *Let  $u$  be a smooth function in  $\Omega_T$ . The asymptotic mean value formula*

$$u(t, x) = \oint_{t-\varepsilon^2}^t \oint_{B_\varepsilon(x)} \psi(x^{-1} \circ y) u(s, y) dy ds + o(\varepsilon^2) \text{ as } \varepsilon \rightarrow 0 \quad (1.11)$$

*holds for all  $(t, x) \in \Omega_T$  if and only if*

$$u_t(t, x) = M(n) \Delta_H u(t, x) \quad (1.12)$$

*in  $\Omega_T$ , where*

$$M(n) = \begin{cases} M_o(n) = \frac{2n+2}{2n+4} \cdot \frac{(n!!)^2}{(n+1)!!(n-1)!!} \cdot \frac{\pi}{2} \cdot \frac{1}{2n}, & \text{if } n \text{ is odd,} \\ M_e(n) = \frac{2n+2}{2n+4} \cdot \frac{(n!!)^2}{(n+1)!!(n-1)!!} \cdot \frac{2}{\pi} \cdot \frac{1}{2n}, & \text{if } n \text{ is even.} \end{cases} \quad (1.13)$$

Next, we study the homogeneous sub-infinity Laplace parabolic equation

$$u_t = \Delta_H^\infty u = |\nabla_0 u|^{-2} \langle (X^2 u)^* \nabla_0 u, \nabla_0 u \rangle. \quad (1.14)$$

Since the right-hand side of equation (1.14) cannot be in a divergence form, we are not able to define a distributional weak solution. However, there is a standard way to define viscosity solutions for singular parabolic equations. We recall this definition follow Evans and Spruck [15], Chen, Giga and Goto [8], Ohnuma and Sato [30], etc. In addition, the homogenous sub-infinity Laplace equation

$$|\nabla_0 u|^{-2} \langle (X^2 u)^* \nabla_0 u, \nabla_0 u \rangle = 0$$

is different from the inhomogeneous sub-infinity Laplace equation

$$\langle (X^2 u)^* \nabla_0 u, \nabla_0 u \rangle = 0,$$

which was studied by Bieske [2] and Wang [32]. The primary difficulty arising from the homogenous sub-infinity Laplace equation will be to modify the theory to cover the possibility that the spatial horizontal gradient  $\nabla_0 u$  may vanish.

For a symmetric matrix  $A$ , we denote its largest and smallest eigenvalue by  $\lambda_{\max}(A)$  and  $\lambda_{\min}(A)$ , respectively. We give a definition of viscosity solutions to equation (1.14) as follows:

**Definition 1.1.** *A lower semi-continuous function  $u : \Omega_T \rightarrow \mathbb{R} \cup \{+\infty\}$  is a viscosity super-solution to (1.14) if for every  $(t^0, x^0) \in \Omega_T$  and  $\phi \in C_H^2(\Omega_T)$  satisfy the following*

- (i)  *$u$  is not identically infinity in each component of  $\Omega_T$ ,*
  - (ii)  *$u(t^0, x^0) = \phi(t^0, x^0)$ , and  $u(t, x) > \phi(t, x)$  for  $(t, x) \neq (t^0, x^0)$ ,*
- then we have at the point  $(t^0, x^0)$*

$$\begin{cases} \phi_t \geq \Delta_H^\infty \phi & \text{if } \nabla_0 \phi(t^0, x^0) \neq 0, \\ \phi_t \geq \lambda_{\min}((X^2 \phi)^*) & \text{if } \nabla_0 \phi(t^0, x^0) = 0. \end{cases}$$

*A function  $u$  is a viscosity sub-solution to (1.14) if  $-u$  is a viscosity super-solution. A function  $u$  is a viscosity solution if it is both a viscosity super-solution and a viscosity sub-solution.*

Similarly to the case in [28], the asymptotic mean value formulae hold in a viscosity sense. We recall the following definition [28].

**Definition 1.2.** A continuous function  $u$  satisfies

$$u(t, x) = \frac{\alpha}{2} \int_{t-\varepsilon^2}^t \left( \max_{y \in \bar{B}_\varepsilon(x)} u(s, y) + \min_{y \in \bar{B}_\varepsilon(x)} u(s, y) \right) ds + \beta \int_{t-\varepsilon^2}^t \int_{B_\varepsilon(x)} \psi(x^{-1} \circ y) u(s, y) ds dy + o(\varepsilon^2) \quad (1.15)$$

as  $\varepsilon \rightarrow 0$  in the viscosity sense if for every  $\phi \in C_H^2$  such that  $u - \phi$  has a strict minimum at the point  $(x, t) \in \Omega_T$  with  $u(x, t) = \phi(x, t)$ , we have

$$\phi(t, x) \geq \frac{\alpha}{2} \int_{t-\varepsilon^2}^t \left( \max_{y \in \bar{B}_\varepsilon(x)} \phi(s, y) + \min_{y \in \bar{B}_\varepsilon(x)} \phi(s, y) \right) ds + \beta \int_{t-\varepsilon^2}^t \int_{B_\varepsilon(x)} \psi(x^{-1} \circ y) \phi(s, y) ds dy + o(\varepsilon^2) \quad (1.16)$$

as  $\varepsilon \rightarrow 0$ , and analogously when testing from above.

Observe that a  $C_H^2$  function ( see Definition 2.1)  $u$  satisfies an equality in the classical sense if and only if it satisfies in the viscosity sense.

**Theorem 1.2.** Let  $u$  be a continuous function in  $\Omega_T$ . The asymptotic mean value formula

$$u(t, x) = \frac{1}{2} \int_{t-\varepsilon^2}^t \left( \max_{y \in \bar{B}_\varepsilon(x)} u(s, y) + \min_{y \in \bar{B}_\varepsilon(x)} u(s, y) \right) ds + o(\varepsilon^2) \quad \text{as } \varepsilon \rightarrow 0 \quad (1.17)$$

holds for all  $(t, x) \in \Omega_T$  in the viscosity sense if and only if  $u$  is a viscosity solution to (1.14).

Finally, we combine the above results to obtain an asymptotic mean value formula of sub- $p$ -Laplace parabolic equations. Recalling the following definition of viscosity solutions.

**Definition 1.3.** A lower semi-continuous function  $u : \Omega_T \rightarrow \mathbb{R} \cup \{+\infty\}$  is a viscosity super-solution to (1.9) if for every  $(t^0, x^0) \in \Omega_T$  and  $\phi \in C_H^2(\Omega_T)$  satisfy the following

- (i)  $u$  is not identically infinity in each component of  $\Omega_T$ ,
  - (ii)  $u(t^0, x^0) = \phi(t^0, x^0)$ , and  $u(t, x) > \phi(t, x)$  for  $(t, x) \neq (t^0, x^0)$ ,
- then we have at the point  $(t^0, x^0)$

$$\begin{cases} \phi_t \geq (p-2)\Delta_H^\infty \phi + \Delta_H \phi & \text{if } \nabla_0 \phi(t^0, x^0) \neq 0, \\ \phi_t \geq \lambda_{\min}((p-2)(X^2 \phi)^*) + \Delta_H \phi & \text{if } \nabla_0 \phi(t^0, x^0) = 0. \end{cases}$$

A function  $u$  is a viscosity sub-solution to (1.9) if  $-u$  is a viscosity super-solution. A function  $u$  is a viscosity solution if it is both a viscosity super-solution and a viscosity sub-solution.

We derive an equivalent definition of the above definition of viscosity solutions by reducing the number of test functions. We will prove that, in the case  $\nabla_0 \phi(t, x) = 0$ , we may assume that  $(X^2 \phi)^*(t, x) = 0$ , and thus  $\lambda_{\min} = \lambda_{\max} = 0$ . Nothing is required if  $\nabla_0 \phi(t, x) \neq 0$  and  $(X^2 \phi)^*(t, x) \neq 0$ . Indeed, we have

**Theorem 1.3.** Suppose  $u : \Omega_T \rightarrow \mathbb{R}$  is a lower semi-continuous function with the property that for every  $(t^0, x^0) \in \Omega_T$  and  $\phi \in C_H^2(\Omega_T)$  satisfying

$$u(t^0, x^0) = \phi(t^0, x^0) \quad \text{and} \quad u(t, x) > \phi(t, x) \quad \text{for } (t, x) \neq (t^0, x^0),$$

the following holds:

$$\begin{cases} \phi_t(t^0, x^0) \geq (p-2)\Delta_H^\infty \phi(t^0, x^0) + \Delta_H \phi(t^0, x^0) & \text{if } \nabla_0 \phi(t^0, x^0) \neq 0, \\ \phi_t(t^0, x^0) \geq 0 & \text{if } \nabla_0 \phi(t^0, x^0) = 0, \text{ and } (X^2 \phi)^*(t^0, x^0) = 0. \end{cases}$$

Then  $u$  is a viscosity super-solution of (1.9). And the same result holds for the viscosity sub-solution.

Theorem 1.1, together with Theorem 1.2, immediately yields the following asymptotic mean value formula of sub- $p$ -Laplace parabolic equations.

**Theorem 1.4.** *Let  $1 < p \leq \infty$  and  $u$  be a continuous function in  $\Omega_T$ . The asymptotic expansion*

$$u(t, x) = \frac{\alpha}{2} \int_{t-\varepsilon^2}^t \left( \max_{y \in \overline{B}_\varepsilon(x)} u(s, y) + \min_{y \in \overline{B}_\varepsilon(x)} u(s, y) \right) ds + \beta \int_{t-\varepsilon^2}^t \int_{B_\varepsilon(x)} \psi(x^{-1} \circ y) u(s, y) ds dy + o(\varepsilon^2) \text{ as } \varepsilon \rightarrow 0$$

*holds for all  $(t, x) \in \Omega_T$  in the viscosity sense if and only if  $u$  is a viscosity solution to*

$$u_t(t, x) = \frac{M(n)}{M(n)(p-2)+1} |\nabla_0 u|^{2-p} \Delta_H^p u(t, x),$$

where  $M(n)$  is as in (1.13), and  $\alpha$  and  $\beta$  satisfy

$$\begin{cases} \beta M(n)(p-2) = \alpha, \\ \alpha + \beta = 1. \end{cases} \quad (1.18)$$

**Remark.** *If  $p = 2$ , then  $\alpha = 0$  and  $\beta = 1$ , and if  $p = \infty$ , then  $\alpha = 1$  and  $\beta = 0$ .*

The rest of the paper is organized as follows. In Section 2 we collect some definitions and results about sub-parabolic jets on  $\mathbb{H}^n$ . Using the polar coordinates on  $\mathbb{H}^n$ , we compute some integrals. By twisting the Euclidean jets to sub-parabolic jets and using the Crandall-Ishii-Lions maximum principle, Theorem 1.3 is proved in Section 3. In Section 4 we prove asymptotic mean value formulae of sub-heat equations, sub-infinity Laplace parabolic equation and sub- $p$ -Laplace parabolic equation, respectively. An example is constructed to show that these formulae do not hold in non-asymptotic sense.

## 2 Sub-parabolic jets and polar coordinates on $\mathbb{H}^n$

In this section, we collect some definitions and results about sub-parabolic jets on  $\mathbb{H}^n$ , and recall the polar coordinates on  $\mathbb{H}^n$ .

**Definition 2.1.** ([16]) *Let  $f : \mathbb{H}^n \rightarrow \mathbb{R}$ , we say that  $f \in C_H^1$ , if  $X_i f$  exists and is continuous at each point of  $\mathbb{H}^n$ , for every  $i = 1, \dots, 2n$ . Moreover, for any nonnegative integer  $m$ , we say that  $f \in C_H^m$ , if  $X^\alpha f$  exists and is continuous at each point of  $\mathbb{H}^n$ , for every horizontal derivation  $X^\alpha = X_{i_1}^{\alpha_1} \dots X_{i_n}^{\alpha_n} X_{i_{n+1}}^{\alpha_{n+1}} \dots X_{i_{2n}}^{\alpha_{2n}}$  with  $0 \leq |\alpha| = \alpha_1 + \dots + \alpha_{2n} \leq m$ .*

Let  $\mathcal{S}^n$  be the set of all real  $n \times n$  symmetric matrixes, we introduce definitions about sub-parabolic jets on  $\mathbb{H}^n$ , which are natural extensions of sub-elliptic jets [2].

**Definition 2.2.** *Let  $u : \Omega_T \rightarrow \mathbb{R}$  be an upper-semicontinuous function. The second order sub-parabolic super-jet of  $u$  at  $(t^0, x^0)$  is defined as*

$$\begin{aligned} \mathcal{J}^{2,+}(u, (t^0, x^0)) &= \{(a, p, Y) \in \mathbb{R} \times \mathbb{R}^{2n+1} \times \mathcal{S}^{2n} \text{ such that} \\ u(t, x) &\leq u(t^0, x^0) + a(t - t^0) + \langle p, ((x^0)^{-1} \circ x) \rangle \\ &+ \frac{1}{2} \langle Y((x^0)^{-1} \circ x), ((x^0)^{-1} \circ x) \rangle + c(|t - t^0| + \rho^2((x^0)^{-1} \circ x))\}. \end{aligned}$$

Similarly, for a lower-semicontinuous function  $u$ , we define the second order sub-parabolic sub-jet

$$\begin{aligned} \mathcal{J}^{2,-}(u, (t^0, x^0)) &= \{(a, p, Y) \in \mathbb{R} \times \mathbb{R}^{2n+1} \times \mathcal{S}^{2n} \text{ such that} \\ u(t, x) &\geq u(t^0, x^0) + a(t - t^0) + \langle p, ((x^0)^{-1} \circ x) \rangle \\ &+ \frac{1}{2} \langle Y((x^0)^{-1} \circ x), ((x^0)^{-1} \circ x) \rangle + c(|t - t^0| + \rho^2((x^0)^{-1} \circ x))\}. \end{aligned}$$

The closures of the jets is defined in the obvious way:

$$\begin{aligned} \overline{\mathcal{J}^{2,+}}(u, (t^0, x^0)) &= \{(a, p, Y) \in \mathbb{R} \times \mathbb{R}^{2n+1} \times \mathcal{S}^{2n} : \exists (t^\alpha, x^\alpha, a^\alpha, p^\alpha, Y^\alpha) \in \Omega_T \times \mathbb{R} \times \mathbb{R}^{2n+1} \times \mathcal{S}^{2n} \\ &\text{such that } (a^\alpha, p^\alpha, Y^\alpha) \in \mathcal{J}^{2,+}(u, (t^\alpha, x^\alpha)) \text{ and } (t^\alpha, x^\alpha, a^\alpha, p^\alpha, Y^\alpha) \rightarrow (t^0, x^0, a, p, Y)\}, \end{aligned}$$

and similarly for  $\overline{\mathcal{J}^{2,-}}$ .

The following proposition characterizes the sub-parabolic jets in terms of test functions that touch from above or below. This proposition is an natural extension of the sub-elliptic case [2].

**Proposition 2.1.** *Define the set*

$$\begin{aligned} K^{2,+}(u, (t^0, x^0)) &= \{(\phi_t(t^0, x^0), \nabla \phi(t^0, x^0), (X^2 \phi)^*(t^0, x^0)) : \phi \in C_H^2(\Omega_T) \text{ and} \\ &u - \phi \text{ has a strict maximum at } (t^0, x^0)\}, \end{aligned}$$

and

$$\begin{aligned} K^{2,-}(u, (t^0, x^0)) &= \{(\phi_t(t^0, x^0), \nabla \phi(t^0, x^0), (X^2 \phi)^*(t^0, x^0)) : \phi \in C_H^2(\Omega_T) \text{ and} \\ &u - \phi \text{ has a strict minimum at } (t^0, x^0)\}. \end{aligned}$$

Then, we have

$$\mathcal{J}^{2,+}(u, (t^0, x^0)) = K^{2,+}(u, (t^0, x^0)),$$

and

$$\mathcal{J}^{2,-}(u, (t^0, x^0)) = K^{2,-}(u, (t^0, x^0)).$$

At the last of this section, we recall polar coordinates on  $\mathbb{H}^n$ , which were introduced for  $\mathbb{H}^1$  by [19] and then extended by Dunkl [14] to  $\mathbb{H}^n$ . Let

$$\begin{cases} x_1 = \rho \sin^{1/2} \phi \sin \theta_1 \cdots \sin \theta_{2n-2} \sin \theta_{2n-1}, \\ x_{n+1} = \rho \sin^{1/2} \phi \sin \theta_1 \cdots \sin \theta_{2n-2} \cos \theta_{2n-1}, \\ x_2 = \rho \sin^{1/2} \phi \sin \theta_1 \cdots \sin \theta_{2n-3} \cos \theta_{2n-2}, \\ x_{n+2} = \rho \sin^{1/2} \phi \sin \theta_1 \cdots \sin \theta_{2n-4} \cos \theta_{2n-3}, \\ \vdots \\ x_n = \rho \sin^{1/2} \phi \sin \theta_1 \cos \theta_2, \\ x_{2n} = \rho \sin^{1/2} \phi \cos \theta_1, \\ x_{2n+1} = \rho^2 \cos \phi. \end{cases} \quad (2.1)$$

Here  $0 \leq \phi < \pi$ ,  $0 \leq \theta_i < \pi$ ,  $i = 1, \dots, 2n-2$  and  $0 \leq \theta_{2n-1} < 2\pi$ . Let  $r = |\bar{x}| = \left(\sum_{i=1}^{2n} x_i^2\right)^{1/2}$ , from (2.1) we get

$$r = |\bar{x}| = \rho \sin^{1/2} \phi. \quad (2.2)$$

By the usual spherical coordinates in  $\mathbb{R}^{2n}$ , we have

$$d\bar{x} = r^{2n-1} dr d\omega,$$

where  $d\omega$  denotes the Lebesgue measure on  $S^{2n-1}$ . From (2.1) and (2.2) we have

$$dr dt = \rho^2 \sin^{-1/2} \phi dp d\phi.$$

Therefore, the Jacobi of (2.1) is

$$\begin{aligned} dx &= \rho^{2n+1} (\sin \phi)^{n-1} dp d\phi dw \\ &= \rho^{2n+1} (\sin \phi)^{n-1} \sin^{2n-2} \theta_1 \cdots \sin \theta_{2n-2} dp d\phi d\theta_1 \cdots d\theta_{2n-1}. \end{aligned} \quad (2.3)$$

Using the the polar coordinates on  $\mathbb{H}^n$ , we calculate to obtain the following Lemma.

**Lemma 2.2.**

$$\int_{B_\varepsilon(x)} \psi(x^{-1} \circ y)(y_i - x_i) dy = 0, \quad i = 1, \dots, 2n,$$

$$\int_{B_\varepsilon(x)} \psi(x^{-1} \circ y) \left( y_{2n+1} - x_{2n+1} + 2 \sum_{i=1}^n (x_i y_{n+i} - x_{n+i} y_i) \right) dy = 0,$$

and

$$\int_{B_\varepsilon(x)} \psi(x^{-1} \circ y)(y_i - x_i) \cdot (y_j - x_j) dy = 0 \quad \text{for } i, j = 1, \dots, 2n, i \neq j.$$

For every  $i = 1, \dots, 2n$ , if  $n$  is even,

$$\int_{B_\varepsilon(x)} \psi(x^{-1} \circ y)(y_i - x_i)^2 dy = \frac{2n+2}{2n+4} \cdot \frac{(n!)^2}{(n+1)!!(n-1)!!} \cdot \frac{1}{2n} \cdot \frac{2}{\pi} \varepsilon^2 \equiv M_e(n) \varepsilon^2,$$

and if  $n$  is odd,

$$\int_{B_\varepsilon(x)} \psi(x^{-1} \circ y)(y_i - x_i)^2 dy = \frac{2n+2}{2n+4} \cdot \frac{(n!)^2}{(n+1)!!(n-1)!!} \cdot \frac{1}{2n} \cdot \frac{\pi}{2} \varepsilon^2 \equiv M_o(n) \varepsilon^2.$$

**Proof** The first three terms are obviously. Using left-invariance and symmetry, we have

$$\int_{B_\varepsilon(x)} \psi(x^{-1} \circ y)(y_i - x_i)^2 dy = \frac{1}{2n} \int_{B_\varepsilon(0)} \psi(y) |\bar{y}|^2 dy.$$

By using (2.1) and (2.3)

$$\begin{aligned} \int_{B_\varepsilon(0)} \psi(y) |\bar{y}|^2 dy &= \frac{\int_0^\varepsilon \rho^{2n+3} d\rho \int_0^\pi \sin^{n+1} \phi d\phi}{\int_0^\varepsilon \rho^{2n+1} d\rho \int_0^\pi \sin^n \phi d\phi} \\ &= \frac{2n+2}{2n+4} \varepsilon^2 \frac{\int_0^\pi \sin^{n+1} \phi d\phi}{\int_0^\pi \sin^n \phi d\phi}. \end{aligned}$$

According to the integrals

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \begin{cases} \frac{(2k-1)!!}{(2k)!!} \frac{\pi}{2}, & n = 2k, \\ \frac{(2k)!!}{(2k+1)!!}, & n = 2k+1, \end{cases}$$

we obtain the desired results in this lemma.  $\square$

### 3 Proof of Theorem 1.3

The general approach for the proof of Theorem 1.3 is similar to [15]; see also [24], [28]. However, we notice that, the Crandall-Ishii-Lions maximum principle (see Theorem 3.2 in [10]) is not available for sub-parabolic structure on the Heisenberg group. To circumvent this, one may use the Euclidean Crandall-Ishii-Lions maximum principle to get the Euclidean jets, and then twist the Euclidean jets to form sub-parabolic jets on  $\mathbb{H}^n$ . This method was introduced by Bieske [2] for studying existence and uniqueness of the viscosity solutions to the sub-infinite Laplace equations on  $\mathbb{H}^1$ .

**Lemma 3.1.** ([2]) Let  $(a, p, Y) \in \mathbb{R} \times \mathbb{R}^{2n+1} \times \mathcal{S}^{2n+1}$ , and  $\|\cdot\|_E$  denote the standard norm in  $\mathbb{R}^{2n+1}$ . Define the standard Euclidean super-jet, denoted by  $\mathcal{J}_E^{2,+}$ ,

$$\begin{aligned} \mathcal{J}_E^{2,+}(u, (t^0, x^0)) &= \left\{ (a, p, Y) : u(t, x) \leq u(t^0, x^0) + a(t - t^0) + \langle p, x - x^0 \rangle \right. \\ &\quad \left. + \frac{1}{2} \langle Y(x - x^0), (x - x^0) \rangle + o(|t - t^0| + \|x - x^0\|_E^2) \right\}. \end{aligned} \quad (3.1)$$



denote

$$A(x^0) = \begin{pmatrix} 1 \cdots 0 & 0 \cdots 0 & 2x_{n+1}^0 \\ \vdots & \vdots & \vdots \\ 0 \cdots 1 & 0 \cdots 0 & 2x_{2n}^0 \\ 0 \cdots 0 & 1 \cdots 0 & -2x_1^0 \\ \vdots & \vdots & \vdots \\ 0 \cdots 0 & 0 \cdots 1 & -2x_n^0 \\ 0 \cdots 0 & 0 \cdots 0 & 1 \end{pmatrix}, \quad (3.2)$$

Then

$$(a, p, Y) \in \mathcal{J}_E^{2,+}(u, (t^0, x^0))$$

implies

$$(a, A(x^0) \cdot p, (A \cdot Y \cdot A^T)_{2n}) \in \mathcal{J}^{2,+}(u, (t^0, x^0))$$

with the convention that for any matrix  $M$ ,  $M_{2n}$  is the  $2n \times 2n$  principal minor.

Now we are in a position to prove Theorem 1.3.

*Proof of Theorem 1.3* Suppose  $u$  is not a viscosity super-solution of (1.9) in the sense of Definition 1.3, but satisfies the assumptions of Theorem 1.3. Then there exist  $(t^0, x^0) \in \Omega_T$  and  $\phi \in C_H^2(\Omega_T)$  satisfying  $u(t^0, x^0) = \phi(t^0, x^0)$ , and  $u(t, x) > \phi(t, x)$  for  $(t, x) \neq (t^0, x^0)$ , for which  $\nabla_0 \phi(t^0, x^0) = 0$ ,  $(X^2 \phi)^*(t^0, x^0) \neq 0$ , and

$$\phi_t(t^0, x^0) < \lambda_{\min}((p-2)(X^2 \phi)^*(t^0, x^0)) + \Delta_H \phi(t^0, x^0). \quad (3.3)$$

Let

$$w^\alpha(t, x, s, y) = u(t, x) - \phi(s, y) + \varphi(t, x, s, y), \quad (3.4)$$

where

$$\varphi(t, x, s, y) = \frac{\alpha}{4} \left( \sum_{i=1}^{2n} (x_i - y_i)^4 + \left( x_{2n+1} - y_{2n+1} + 2 \sum_{i=1}^n (x_{n+i} y_i - x_i y_{n+i}) \right)^4 \right) + \frac{\alpha}{2} (t - s)^2,$$

and denote by  $(t^\alpha, x^\alpha, s^\alpha, y^\alpha)$  the minimum point of  $w^\alpha$  in  $\overline{\Omega_T} \times \overline{\Omega_T}$ . Since  $(t^0, x^0)$  is a local minimum for  $u - \phi$  and by [2], we may assume that

$$(t^\alpha, x^\alpha, s^\alpha, y^\alpha) \rightarrow (t^0, x^0, t^0, x^0) \quad \text{as } \alpha \rightarrow +\infty.$$

In particular,  $(t^\alpha, x^\alpha) \in \Omega_T$  and  $(s^\alpha, y^\alpha) \in \Omega_T$  for all  $\alpha$  large enough.

We consider two cases: either  $\overline{x^\alpha} = \overline{y^\alpha}$  or  $\overline{x^\alpha} \neq \overline{y^\alpha}$  for all  $\alpha$  large enough.

*Case 1:* Let  $\overline{x^\alpha} = \overline{y^\alpha}$ , and denote

$$\vartheta(s, y) = \varphi(t^\alpha, x^\alpha, s, y). \quad (3.5)$$

Then  $\phi(s, y) - \vartheta(s, y)$  has a local maximum at  $(s^\alpha, y^\alpha)$ , and thus

$$\phi_s(s^\alpha, y^\alpha) = \vartheta_s(s^\alpha, y^\alpha) \text{ and } (X^2 \phi)^*(s^\alpha, y^\alpha) \leq (X^2 \vartheta)^*(s^\alpha, y^\alpha).$$

A direct calculation yields

$$(X^2 \vartheta)^*(s^\alpha, y^\alpha) = 0 \text{ provided } \overline{x^\alpha} = \overline{y^\alpha}, \text{ and } \vartheta_s(s^\alpha, y^\alpha) = -\alpha(t^\alpha - s^\alpha),$$

and thus,

$$\phi_s(s^\alpha, y^\alpha) = -\alpha(t^\alpha - s^\alpha), \text{ and } (X^2 \phi)^*(s^\alpha, y^\alpha) \leq 0. \quad (3.6)$$

By (3.3) and continuity of

$$(t, x) \mapsto \lambda_{\min}((p-2)(X^2 \phi)^*(t, x)) + \Delta_H \phi(t, x),$$

we have

$$\phi_s(s^\alpha, y^\alpha) < \lambda_{\min} \left( (p-2)(X^2\phi)^*(s^\alpha, y^\alpha) \right) + \Delta_H \phi(s^\alpha, y^\alpha) \quad (3.7)$$

for  $\alpha$  large enough. By (3.6) and (3.7), we have

$$0 < -\vartheta_s(s^\alpha, y^\alpha) = \alpha(t^\alpha - s^\alpha), \quad (3.8)$$

for  $\alpha$  large enough. If  $1 < p < 2$ , the inequality follows from the calculation

$$\begin{aligned} & \lambda_{\min} \left( (p-2)(X^2\phi)^*(s^\alpha, y^\alpha) \right) + \Delta_H \phi(s^\alpha, y^\alpha) \\ &= (p-2)\lambda_{\max} \left( (X^2\phi)^*(s^\alpha, y^\alpha) \right) + \text{trace} \left( (X^2\phi)^*(s^\alpha, y^\alpha) \right) \\ &= (p-1)\lambda_{\max} + \sum_{\lambda_i \neq \lambda_{\max}} \lambda_i \leq 0, \end{aligned} \quad (3.9)$$

where  $\lambda_i, \lambda_{\max}$  denote the eigenvalue and the maximum eigenvalue of  $(X^2\phi)^*(s^\alpha, y^\alpha)$ , respectively.

On the other hand, let

$$\mu(t, x) = -\varphi(t, x, s^\alpha, y^\alpha). \quad (3.10)$$

Similarly,  $u(t, x) - \mu(t, x)$  has a local minimum at  $(t^\alpha, x^\alpha)$ , and

$$\nabla_0 \mu(t^\alpha, x^\alpha) = 0, \quad (X^2\mu)^*(t^\alpha, x^\alpha) = 0, \quad \text{provided } \overline{x^\alpha} = \overline{y^\alpha}.$$

That is,  $\mu$  is a  $C_H^2$  test function, by the assumption on  $u$ , we have

$$0 \leq \mu_t(t^\alpha, x^\alpha) = -\alpha(t^\alpha - s^\alpha), \quad (3.11)$$

for  $\alpha$  large enough. Summing up (3.8) and (3.11) gives

$$0 < \alpha(t^\alpha - s^\alpha) - \alpha(t^\alpha - s^\alpha) = 0.$$

This is a contradiction.

*Case 2:* Next we consider the case  $\overline{x^\alpha} \neq \overline{y^\alpha}$  for all  $\alpha$  large enough. We apply the Euclidean maximum principle for semi-continuous functions of Crandall-Ishii-Lions (see Theorem 3.2 in [10]). There exists  $(2n+1) \times (2n+1)$  symmetric matrices  $Y^\alpha, Z^\alpha$  such that

$$\begin{aligned} & \left( -D_s \varphi(t^\alpha, x^\alpha, s^\alpha, y^\alpha), -D_y \varphi(t^\alpha, x^\alpha, s^\alpha, y^\alpha), Y^\alpha \right) \in \overline{\mathcal{F}}_E^{2,+} \phi(s^\alpha, y^\alpha), \\ & \left( D_t \varphi(t^\alpha, x^\alpha, s^\alpha, y^\alpha), D_x \varphi(t^\alpha, x^\alpha, s^\alpha, y^\alpha), Z^\alpha \right) \in \overline{\mathcal{F}}_E^{2,-} u(t^\alpha, x^\alpha). \end{aligned}$$

with the property that

$$\langle Y^\alpha \gamma, \gamma \rangle - \langle Z^\alpha \chi, \chi \rangle \leq \langle C \gamma \oplus \chi, \gamma \oplus \chi \rangle, \quad (3.12)$$

where

$$C = B + \frac{1}{\alpha} B^2, \quad \gamma \oplus \chi = (\gamma, \chi),$$

and

$$B = D_{x,y}^2 \varphi(t^\alpha, x^\alpha, s^\alpha, y^\alpha),$$

with the notations  $D_x, D_y$  and  $D_{xy}$  denote the Euclidean derivatives. By using Lemma 3.1 and the fact

$$-D_s \varphi(t^\alpha, x^\alpha, s^\alpha, y^\alpha) = D_t \varphi(t^\alpha, x^\alpha, s^\alpha, y^\alpha) = \alpha(t^\alpha - s^\alpha),$$

we conclude that

$$\left( \alpha(t^\alpha - s^\alpha), -\nabla_y \varphi(t^\alpha, x^\alpha, s^\alpha, y^\alpha), \widetilde{Y^\alpha} \right) \in \overline{\mathcal{F}}_E^{2,+} \phi(s^\alpha, y^\alpha), \quad (3.13)$$

and

$$\left( \alpha(t^\alpha - s^\alpha), -\nabla_x \varphi(t^\alpha, x^\alpha, s^\alpha, y^\alpha), \widetilde{Z^\alpha} \right) \in \overline{\mathcal{F}}_E^{2,-} u(t^\alpha, x^\alpha), \quad (3.14)$$

where  $\widetilde{Y}^\alpha$  and  $\widetilde{Z}^\alpha$  are  $2n \times 2n$  symmetric matrices defined by

$$\widetilde{Y}^\alpha = \left( A(y^\alpha) \cdot Y^\alpha \cdot A(y^\alpha)^T \right)_{2n}$$

and

$$\widetilde{Z}^\alpha = \left( A(x^\alpha) \cdot Z^\alpha \cdot A(x^\alpha)^T \right)_{2n},$$

where  $A(x^\alpha)$  and  $A(y^\alpha)$  are as in (3.2) with the point  $x^0$  is replaced by  $x^\alpha$  and  $y^\alpha$ , respectively.

*Claim:* Let  $\xi = (y^\alpha)^{-1} \circ x^\alpha \in \mathbb{R}^{2n}$ , we have the following estimate

$$\langle \widetilde{Y}^\alpha \xi, \xi \rangle - \langle \widetilde{Z}^\alpha \xi, \xi \rangle \leq 0, \quad \text{as } \alpha \rightarrow +\infty. \quad (3.15)$$

Assume the above claim is true. By (3.3), there exists a constant  $\theta > 0$ , such that

$$\theta + \phi_t(t^0, x^0) < \lambda_{\min} \left( (p-2)(X^2\phi)^*(t^0, x^0) \right) + \Delta_H \phi(t^0, x^0), \quad (3.16)$$

and with the continuity of

$$(t, x) \mapsto \lambda_{\min} \left( (p-2)(X^2\phi)^*(t, x) \right) + \Delta_H \phi(t, x),$$

we have

$$\theta + \phi_s(s^\alpha, y^\alpha) < \lambda_{\min} \left( (p-2)(X^2\phi)^*(s^\alpha, y^\alpha) \right) + \Delta_H \phi(s^\alpha, y^\alpha) \quad (3.17)$$

for  $\alpha$  large enough.

Using (3.13), (3.14), (3.17) and the the assumptions on  $u$ , we have

$$\begin{aligned} \theta &= \theta + \alpha(t^\alpha - s^\alpha) - \alpha(t^\alpha - s^\alpha) \\ &< (p-2) \left\langle \widetilde{Y}^\alpha \frac{\overline{(y^\alpha)^{-1} \circ x^\alpha}}{|(y^\alpha)^{-1} \circ x^\alpha|}, \frac{\overline{(y^\alpha)^{-1} \circ x^\alpha}}{|(y^\alpha)^{-1} \circ x^\alpha|} \right\rangle + \text{trace}(\widetilde{Y}^\alpha) \\ &\quad - (p-2) \left\langle \widetilde{Z}^\alpha \frac{\overline{(y^\alpha)^{-1} \circ x^\alpha}}{|(y^\alpha)^{-1} \circ x^\alpha|}, \frac{\overline{(y^\alpha)^{-1} \circ x^\alpha}}{|(y^\alpha)^{-1} \circ x^\alpha|} \right\rangle - \text{trace}(\widetilde{Z}^\alpha) \\ &= (p-2) \left\langle (\widetilde{Y}^\alpha - \widetilde{Z}^\alpha) \frac{\overline{(y^\alpha)^{-1} \circ x^\alpha}}{|(y^\alpha)^{-1} \circ x^\alpha|}, \frac{\overline{(y^\alpha)^{-1} \circ x^\alpha}}{|(y^\alpha)^{-1} \circ x^\alpha|} \right\rangle + \text{trace}(\widetilde{Y}^\alpha - \widetilde{Z}^\alpha) \\ &\leq 0, \end{aligned}$$

the last inequality being valid by the claim (3.15) in the case  $p > 2$ . If  $1 < p < 2$ , the last inequality follows the same calculation in (3.9). This is a contradiction.

*Proof of the claim.* For any  $\xi = (\xi_1, \dots, \xi_{2n}) \in \mathbb{R}^{2n}$ , let

$$\zeta = \left( \xi, 2 \sum_{i=1}^n (\xi_i y_{n+i}^\alpha - \xi_{n+i} y_i^\alpha) \right), \quad \eta = \left( \xi, 2 \sum_{i=1}^n (\xi_i x_{n+i}^\alpha - \xi_{n+i} x_i^\alpha) \right). \quad (3.18)$$

Recalling the definitions of  $\widetilde{Y}^\alpha$ ,  $\widetilde{Z}^\alpha$ , and combining (3.12), we obtain

$$\langle \widetilde{Y}^\alpha \xi, \xi \rangle - \langle \widetilde{Z}^\alpha \xi, \xi \rangle = \langle Y^\alpha \zeta, \zeta \rangle - \langle Z^\alpha \eta, \eta \rangle \leq \langle C\zeta \oplus \eta, \zeta \oplus \eta \rangle.$$

Straightforward computations show that

$$\langle B\zeta \oplus \eta, \zeta \oplus \eta \rangle = 0, \quad (3.19)$$

and

$$\langle B^2 \zeta \oplus \eta, \zeta \oplus \eta \rangle = 8\alpha^2 \xi^2 \left( x_{2n+1}^\alpha - y_{2n+1}^\alpha + 2 \sum_{i=1}^n (x_{n+i}^\alpha y_i^\alpha - x_i^\alpha y_{n+i}^\alpha) \right)^6. \quad (3.20)$$

Now choosing  $\xi = \overline{(y^\alpha)^{-1} \circ x^\alpha} = (x_1^\alpha - y_1^\alpha, \dots, x_{2n}^\alpha - y_{2n}^\alpha)$ , and noting that [4]

$$\lim_{\alpha \rightarrow +\infty} \alpha \left( \sum_{i=1}^{2n} (x_i - y_i)^4 + \left( x_{2n+1} - y_{2n+1} + 2 \sum_{i=1}^n (x_{n+i} y_i - x_i y_{n+i}) \right)^4 \right) = 0$$

Thanks to  $C = B + 1/\alpha B^2$ , we have

$$\langle C\xi \oplus \eta, \xi \oplus \eta \rangle \rightarrow 0, \text{ as } \alpha \rightarrow +\infty. \quad (3.21)$$

The claimed (3.15) is proved.  $\square$

## 4 Proof of the main results

In this section, we prove asymptotic mean value formulae for sub-heat equations (i.e.  $p = 2$ ) and sub-infinity Laplace parabolic equations (i.e.  $p = \infty$ ) on  $\mathbb{H}^n$ , and construct an example to show that the formulae do not hold in non-asymptotic sense. We begin with a key lemma, which depicts the directions of horizontal maximum and minima of a function, and whose Euclidean version is obvious (cf. [26]).

For  $\phi \in C_H^2(\Omega)$ ,  $x^0 \in \Omega$  and  $r > 0$  with  $\overline{B_r(x^0)} \subset \Omega$ , we define

$$M(r) = \max_{x \in \partial B_r(x^0)} \phi(x), \text{ and } m(r) = \min_{x \in \partial B_r(x^0)} \phi(x).$$

In addition,  $(x^r)^+ \in \partial B_r(x^0)$  and  $(x^r)^- \in \partial B_r(x^0)$  denote any point such that

$$\phi((x^r)^+) = M(r), \text{ and } \phi((x^r)^-) = m(r).$$

Define the set of horizontal maximum directions of  $\phi$  at  $x^0$  to be the set

$$E^+(x^0) = \left\{ \lim_k \frac{\overline{(x^0)^{-1} \circ (x^{r_k})^+}}{r_k} \text{ for some sequence } r_k \rightarrow 0 \right\},$$

and the set of horizontal minimum directions of  $\phi$  at  $x^0$  to be the set

$$E^-(x^0) = \left\{ \lim_k \frac{\overline{(x^0)^{-1} \circ (x^{r_k})^-}}{r_k} \text{ for some sequence } r_k \rightarrow 0 \right\}.$$

**Lemma 4.1.** *Let  $\phi \in C_H^2$  and  $\nabla_0 \phi(x^0) \neq 0$ , then*

$$E^+(x^0) = \frac{\nabla_0 \phi}{|\nabla_0 \phi|}(x^0), \text{ and } E^-(x^0) = -\frac{\nabla_0 \phi}{|\nabla_0 \phi|}(x^0).$$

**Proof** Define a Lagrange function to be

$$\begin{aligned} F(x) &= \phi(x) + \lambda \left( \rho^4((x^0)^{-1} \circ x) - \varepsilon^4 \right) \\ &= \phi(\bar{x}, x_{2n+1}) + \lambda \left\{ |\bar{x} - \bar{x}^0|^4 + \left( x_{2n+1} - x_{2n+1}^0 + 2 \sum_{i=1}^n (x_i^0 x_{n+i} - x_i x_{n+i}^0) \right)^2 - \varepsilon^4 \right\}. \end{aligned}$$

If  $x^\varepsilon$  is a solution of  $\min_{\partial B_\varepsilon(x^0)} \phi(x)$ , then there exists  $\lambda^\varepsilon$ , such that for  $i = 1, \dots, n$

$$\left\{ \begin{array}{l} 0 = X_i F(x^\varepsilon) \\ \quad = X_i \phi(x^\varepsilon) + 4\lambda^\varepsilon \left\{ \left| \overline{x^\varepsilon} - \overline{x^0} \right|^2 (x_i^\varepsilon - x_i^0) + (x_{2n+1}^\varepsilon - x_{2n+1}^0 + 2 \sum_{i=1}^n (x_i^0 x_{n+i}^\varepsilon - x_i^\varepsilon x_{n+i}^0)) \cdot (x_{n+i}^\varepsilon - x_{n+i}^0) \right\}, \\ 0 = X_{n+i} F(x^\varepsilon) \\ \quad = X_{n+i} \phi(x^\varepsilon) + 4\lambda^\varepsilon \left\{ \left| \overline{x^\varepsilon} - \overline{x^0} \right|^2 (x_{n+i}^\varepsilon - x_{n+i}^0) + (x_{2n+1}^\varepsilon - x_{2n+1}^0 + 2 \sum_{i=1}^n (x_i^0 x_{n+i}^\varepsilon - x_i^\varepsilon x_{n+i}^0)) \cdot (x_i^0 - x_i^\varepsilon) \right\}, \\ 0 = T F(x^\varepsilon) \\ \quad = T \phi(x^\varepsilon) + 2\lambda^\varepsilon (x_{2n+1}^\varepsilon - x_{2n+1}^0 + 2 \sum_{i=1}^n (x_i^0 x_{n+i}^\varepsilon - x_i^\varepsilon x_{n+i}^0)), \\ 0 = \frac{\partial F}{\partial \lambda}(x^\varepsilon) \\ \quad = \left| \overline{x^\varepsilon} - \overline{x^0} \right|^4 + (x_{2n+1}^\varepsilon - x_{2n+1}^0 + 2 \sum_{i=1}^n (x_i^0 x_{n+i}^\varepsilon - x_i^\varepsilon x_{n+i}^0))^2 - \varepsilon^4. \end{array} \right.$$

A direct computation yields

$$|\nabla_0 \phi|(x^\varepsilon) = \sum_{i=1}^{2n} X_i \phi(x^\varepsilon) = 4\lambda^\varepsilon \varepsilon^2 \left| \overline{x^\varepsilon} - \overline{x^0} \right|.$$

Therefore

$$\frac{X_i \phi}{|\nabla_0 \phi|}(x^\varepsilon) = - \frac{\left| \overline{x^\varepsilon} - \overline{x^0} \right|^2 (x_i^\varepsilon - x_i^0) + (x_{2n+1}^\varepsilon - x_{2n+1}^0 + 2 \sum_{i=1}^n (x_i^0 x_{n+i}^\varepsilon - x_i^\varepsilon x_{n+i}^0)) \cdot (x_{n+i}^\varepsilon - x_{n+i}^0)}{\varepsilon^2 \left| \overline{x^\varepsilon} - \overline{x^0} \right|}.$$

Similarly,

$$\frac{X_{n+i} \phi}{|\nabla_0 \phi|}(x^\varepsilon) = - \frac{\left| \overline{x^\varepsilon} - \overline{x^0} \right|^2 (x_{n+i}^\varepsilon - x_{n+i}^0) + (x_{2n+1}^\varepsilon - x_{2n+1}^0 + 2 \sum_{i=1}^n (x_i^0 x_{n+i}^\varepsilon - x_i^\varepsilon x_{n+i}^0)) \cdot (x_i^0 - x_i^\varepsilon)}{\varepsilon^2 \left| \overline{x^\varepsilon} - \overline{x^0} \right|},$$

and

$$\frac{T \phi}{|\nabla_0 \phi|}(x^\varepsilon) = - \frac{x_{2n+1}^\varepsilon - x_{2n+1}^0 + 2 \sum_{i=1}^n (x_i^0 x_{n+i}^\varepsilon - x_i^\varepsilon x_{n+i}^0)}{2\varepsilon^2 \left| \overline{x^\varepsilon} - \overline{x^0} \right|}. \quad (4.1)$$

Let  $\varepsilon \rightarrow 0$  in (4.1), we get

$$\frac{T \phi}{|\nabla_0 \phi|}(x^0) = \lim_{\varepsilon \rightarrow 0} - \frac{x_{2n+1}^\varepsilon - x_{2n+1}^0 + 2 \sum_{i=1}^n (x_i^0 x_{n+i}^\varepsilon - x_i^\varepsilon x_{n+i}^0)}{2\varepsilon^2 \left| \overline{x^\varepsilon} - \overline{x^0} \right|}.$$

Therefore

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{(x_{2n+1}^\varepsilon - x_{2n+1}^0 + 2 \sum_{i=1}^n (x_i^0 x_{n+i}^\varepsilon - x_i^\varepsilon x_{n+i}^0)) (x_{n+i}^\varepsilon - x_{n+i}^0)}{\varepsilon^2 \left| \overline{x^\varepsilon} - \overline{x^0} \right|} &= 0, \\ \lim_{\varepsilon \rightarrow 0} \frac{(x_{2n+1}^\varepsilon - x_{2n+1}^0 + 2 \sum_{i=1}^n (x_i^0 x_{n+i}^\varepsilon - x_i^\varepsilon x_{n+i}^0)) (x_i^0 - x_i^\varepsilon)}{\varepsilon^2 \left| \overline{x^\varepsilon} - \overline{x^0} \right|} &= 0, \end{aligned}$$

and

$$\lim_{\varepsilon \rightarrow 0} \frac{\left| \overline{x^\varepsilon} - \overline{x^0} \right|}{\varepsilon} = 1.$$

Hence

$$-\frac{X_i \phi}{|X \phi|}(x^0) = \lim_{\varepsilon \rightarrow 0} \frac{x_i^\varepsilon - x_i^0}{\varepsilon} \cdot \lim_{\varepsilon \rightarrow 0} \frac{\left| \overline{x^\varepsilon} - \overline{x^0} \right|}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{x_i^\varepsilon - x_i^0}{\varepsilon},$$

and

$$-\frac{X_{n+i}\phi}{|X\phi|}(x^0) = \lim_{\varepsilon \rightarrow 0} \frac{x_{n+i}^\varepsilon - x_{n+i}^0}{\varepsilon} \cdot \lim_{\varepsilon \rightarrow 0} \frac{|\overline{x^\varepsilon} - \overline{x^0}|}{\varepsilon} = \lim_{\varepsilon \rightarrow 0} \frac{x_{n+i}^\varepsilon - x_{n+i}^0}{\varepsilon}.$$

That is  $E^-(x^0) = -\frac{X\phi}{|\nabla_0\phi|}(x^0)$ . The same argument to show  $E^+(x^0) = \frac{X\phi}{|\nabla_0\phi|}(x^0)$ . Therefore, the proof of the lemma is complete.  $\square$

Now, we prove an asymptotic mean value formula of the sub-heat equations on  $\mathbb{H}^n$ .

*Proof of Theorem 1.1:* Let  $u$  be a smooth function, and  $(t, x) \in \Omega_T$ . Consider the Taylor expansion

$$\begin{aligned} u(s, y) &= u(t, x) + u_t(t, x)(s - t) + \nabla u(t, x) \cdot (x^{-1} \circ y) \\ &\quad + \frac{1}{2} \left\langle (X^2 u)^*(t, x) \overline{(x^{-1} \circ y)}, \overline{(x^{-1} \circ y)} \right\rangle + c \left( \rho^2(x^{-1} \circ y) + |s - t| \right). \end{aligned} \quad (4.2)$$

Averaging both sides of (4.2), we have

$$\begin{aligned} &\int_{t-\varepsilon^2}^t \int_{B_\varepsilon(x)} \psi(x^{-1} \circ y) u(s, y) dy ds \\ &= u(t, x) + u_t(t, x) \int_{t-\varepsilon^2}^t (s - t) ds + \int_{B_\varepsilon(x)} \psi(x^{-1} \circ y) \nabla u(t, x) \cdot (x^{-1} \circ y) dy \\ &\quad + \frac{1}{2} \int_{B_\varepsilon(x)} \psi(x^{-1} \circ y) \left\langle (X^2 u)^*(t, x) \overline{(x^{-1} \circ y)}, \overline{(x^{-1} \circ y)} \right\rangle dy + o(\varepsilon^2). \end{aligned} \quad (4.3)$$

By Lemma 2.2, we get

$$\int_{B_\varepsilon(x)} \psi(x^{-1} \circ y) \nabla u(t, x) \cdot (x^{-1} \circ y) dy = 0, \quad (4.4)$$

and

$$\frac{1}{2} \int_{B_\varepsilon(x)} \psi(x^{-1} \circ y) \left\langle (X^2 u)^*(t, x) \overline{(x^{-1} \circ y)}, \overline{(x^{-1} \circ y)} \right\rangle dy = \frac{1}{2} M(n) \varepsilon^2 \Delta_H u(t, x). \quad (4.5)$$

Finally,

$$\int_{t-\varepsilon^2}^t (s - t) ds = -\frac{1}{2} \varepsilon^2. \quad (4.6)$$

Substituting (4.4), (4.5) and (4.6) into (4.3), we have

$$\int_{t-\varepsilon^2}^t \int_{B_\varepsilon(x)} \psi(x^{-1} \circ y) u(s, y) dy ds = u(t, x) + \frac{1}{2} \varepsilon^2 \left( M(n) \Delta_H u(t, x) - u_t(t, x) \right) + o(\varepsilon^2). \quad (4.7)$$

This holds for any smooth function.

We first prove that if  $u$  satisfies the asymptotic mean value formula (1.11), then  $u$  is a solution to (1.12). By (4.7), we have

$$\begin{aligned} u(t, x) &= \int_{t-\varepsilon^2}^t \int_{B_\varepsilon(x)} \psi(x^{-1} \circ y) u(s, y) dy ds + o(\varepsilon^2) \\ &= u(t, x) + \frac{1}{2} \varepsilon^2 \left( M(n) \Delta_H u(t, x) - u_t(t, x) \right) + o(\varepsilon^2). \end{aligned}$$

That is

$$\frac{1}{2} \varepsilon^2 \left( M(n) \Delta_H u(t, x) - u_t(t, x) \right) + o(\varepsilon^2) = 0. \quad (4.8)$$

Dividing (4.8) by  $\varepsilon^2$  and passing to the limit  $\varepsilon \rightarrow 0$ , we have

$$u_t(t, x) = M(n) \Delta_H u(t, x). \quad (4.9)$$

Next we are ready to prove the converse implication. If  $u$  is a solution of (1.12), then (4.7) implies that

$$u(t, x) = \int_{t-\varepsilon^2}^t \int_{B_\varepsilon(x)} \psi(x^{-1} \circ y) u(s, y) dy ds + o(\varepsilon^2).$$

This ends the proof.  $\square$

The same argument shows that solutions to the sub-heat equation

$$u_t(t, x) = \Delta_H u(t, x),$$

are characterized by the asymptotic mean value formula

$$u(t, x) = \int_{t-M(n)\varepsilon^2}^t \int_{B_\varepsilon(x)} \psi(x^{-1} \circ y) u(s, y) dy ds + o(\varepsilon^2) \quad \text{as } \varepsilon \rightarrow 0. \quad (4.10)$$

Consider the mean value formula (1.7) for  $H$ -harmonic functions on  $\mathbb{H}^n$ , it is natural to ask if the formula (4.10) holds in a non-asymptotic sense. To be more precise, if  $u$  is a solution to

$$u_t(t, x) = \Delta_H u(t, x),$$

does the equation

$$u(t, x) = \int_{t-M(n)\varepsilon^2}^t \int_{B_\varepsilon(x)} \psi(x^{-1} \circ y) u(s, y) dy ds$$

hold at all  $(t, x) \in \Omega_T$  for all  $\varepsilon > 0$  enough small. The answer to this question is negative, we give an example as follows.

Let

$$u(t, x) = 12t^2 + 12x_1^2 t + x_1^4,$$

where  $x = (x_1, x_2, x_3) \in \mathbb{H}^1$ . It is easy to check that  $u$  is a solution of

$$u_t(t, x) = \Delta_H u(t, x).$$

A direct calculation yields  $M(1) = \frac{\pi}{12}$ , and

$$\int_{B_\varepsilon(0)} \psi(y) dy = \pi \varepsilon^4.$$

Thus

$$\begin{aligned} \int_{B_\varepsilon(0)} \psi(y) u(s, y) dy &= \int_{B_\varepsilon(0)} \psi(y) (12s^2 + 12y_1^2 s + y_1^4) dy \\ &= 12s^2 + \pi \varepsilon^2 s + \frac{1}{8} \varepsilon^4, \end{aligned}$$

and

$$\begin{aligned} &\int_{1-\frac{\pi}{12}\varepsilon^2}^1 (12s^2 + \pi \varepsilon^2 s + \frac{1}{8} \varepsilon^4) ds \\ &= 12 - \pi \varepsilon^2 + \frac{1}{8} \varepsilon^4 + \pi \varepsilon^4 + \frac{1}{36} \pi^2 \varepsilon^4 - \frac{1}{24} \pi^2 \varepsilon^6. \end{aligned}$$

That is

$$\int_{1-\frac{\pi}{12}\varepsilon^2}^1 \int_{B_\varepsilon(0)} \psi(y) u(y, s) dy ds \neq u(0, 1) = 12.$$

Next, we characterize the viscosity solutions of the homogeneous sub-infinity Laplace parabolic equation in terms of an asymptotic mean value formula on  $\mathbb{H}^n$ .

*Proof of Theorem 1.2* Choose a point  $(t, x) \in \Omega_T$ ,  $\varepsilon > 0$ ,  $s \in (t - \varepsilon^2, t)$  and any  $\phi \in C_H^2(\Omega_T)$ . Denote by  $x^{\varepsilon, s}$  be a point at which  $\phi$  attains its minimum in  $\overline{B_\varepsilon(x)}$  at time  $s$ , that is

$$\phi(s, x^{\varepsilon, s}) = \min_{y \in \overline{B_\varepsilon(x)}} \phi(s, y).$$

Consider the Taylor expansion

$$\begin{aligned} \phi(s, y) &= \phi(t, x) + \phi_t(t, x)(s - t) + \nabla \phi(t, x) \cdot (x^{-1} \circ y) \\ &\quad + \frac{1}{2} \left\langle (X^2 \phi)^*(t, x) \overline{(x^{-1} \circ y)}, \overline{(x^{-1} \circ y)} \right\rangle + c \left( \rho^2(x^{-1} \circ y) + |s - t| \right). \end{aligned} \quad (4.11)$$

Taking  $y = x^{\varepsilon, s}$  in (4.11) and noting

$$x^{-1} \circ x^{\varepsilon, s} = \left( x_1^{\varepsilon, s} - x_1, \dots, x_{2n}^{\varepsilon, s} - x_{2n}, x_{2n+1}^{\varepsilon, s} - x_{2n+1} + 2 \sum_{i=1}^n (x_{n+i}^{\varepsilon, s} x_i - x_i^{\varepsilon, s} x_{n+i}) \right),$$

we have

$$\begin{aligned} \phi(s, x^{\varepsilon, s}) &= \phi(t, x) + \phi_t(t, x)(s - t) + \nabla \phi(t, x) \cdot (x^{-1} \circ x^{\varepsilon, s}) \\ &\quad + \frac{1}{2} \left\langle (X^2 \phi)^*(t, x) \overline{(x^{-1} \circ x^{\varepsilon, s})}, \overline{(x^{-1} \circ x^{\varepsilon, s})} \right\rangle + c \left( \varepsilon^2 + |s - t| \right) \\ &= \phi(t, x) + \phi_t(t, x)(s - t) + \sum_{i=1}^{2n} X_i \phi(t, x) (x_i^{\varepsilon, s} - x_i) \\ &\quad + T \phi(t, x) \left( x_{2n+1}^{\varepsilon, s} - x_{2n+1} + 2 \sum_{i=1}^n (x_{n+i}^{\varepsilon, s} x_i - x_i^{\varepsilon, s} x_{n+i}) \right) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^{2n} X_i X_j \phi(t, x) \cdot (x_i^{\varepsilon, s} - x_i) \cdot (x_j^{\varepsilon, s} - x_j) + c \left( \varepsilon^2 + |s - t| \right) \quad \text{as } \varepsilon \rightarrow 0. \end{aligned} \quad (4.12)$$

Similarly, taking  $y = y^{\varepsilon, s} = (2x_1 - x_1^{\varepsilon, s}, \dots, 2x_{2n} - x_{2n}^{\varepsilon, s}, 2x_{2n+1} - x_{2n+1}^{\varepsilon, s})$  in (4.11), and

$$(x)^{-1} \circ y^{\varepsilon, s} = \left( x_1 - x_1^{\varepsilon, s}, \dots, x_{2n} - x_{2n}^{\varepsilon, s}, x_{2n+1} - x_{2n+1}^{\varepsilon, s} + 2 \sum_{i=1}^n (x_i^{\varepsilon, s} x_{n+i} - x_i x_{n+i}^{\varepsilon, s}) \right),$$

we have

$$\begin{aligned} \phi(s, y^{\varepsilon, s}) &= \phi(t, x) + \phi_t(t, x)(s - t) - \sum_{i=1}^{2n} X_i \phi(t, x) (x_i^{\varepsilon, s} - x_i) \\ &\quad - T \phi(t, x) \left( x_{2n+1}^{\varepsilon, s} - x_{2n+1} + 2 \sum_{i=1}^n (x_{n+i}^{\varepsilon, s} x_i - x_i^{\varepsilon, s} x_{n+i}) \right) \\ &\quad + \frac{1}{2} \sum_{i,j=1}^{2n} X_i X_j \phi(t, x) \cdot (x_i^{\varepsilon, s} - x_i) \cdot (x_j^{\varepsilon, s} - x_j) + c \left( \varepsilon^2 + |s - t| \right). \end{aligned} \quad (4.13)$$

Summing (4.12) and (4.13), we have

$$\begin{aligned} &\phi(s, x^{\varepsilon, s}) + \phi(s, y^{\varepsilon, s}) - 2\phi(t, x) \\ &= 2\phi_t(t, x)(s - t) + \sum_{i,j=1}^{2n} X_i X_j \phi(t, x) \cdot (x_i^{\varepsilon, s} - x_i) \cdot (x_j^{\varepsilon, s} - x_j) + o \left( \varepsilon^2 + |s - t| \right). \end{aligned}$$

Since  $x^{\varepsilon, s}$  is a minimum point of  $\phi(\cdot, s)$  on  $\overline{B_\varepsilon(x)}$ , we get

$$\phi(s, x^{\varepsilon, s}) + \phi(s, y^{\varepsilon, s}) - 2\phi(t, x) \leq \max_{y \in \overline{B_\varepsilon(x)}} u(s, y) + \min_{y \in \overline{B_\varepsilon(x)}} u(s, y) - 2\phi(t, x),$$



and thus

$$\begin{aligned} & \max_{y \in \overline{B_\varepsilon}(x)} u(s, y) + \min_{y \in \overline{B_\varepsilon}(x)} u(s, y) - 2\phi(t, x) \\ & \geq 2\phi_t(x, t)(s - t) + \sum_{i,j=1}^{2n} X_i X_j \phi(t, x) \cdot (x_i^{\varepsilon,s} - x_i) \cdot (x_j^{\varepsilon,s} - x_j) + o(\varepsilon^2 + |s - t|). \end{aligned}$$

Integration over the time interval and the fact  $\int_{t-\varepsilon^2}^t (s - t) ds = -\frac{1}{2}\varepsilon^2$  imply

$$\begin{aligned} & \frac{1}{2} \int_{t-\varepsilon^2}^t \left( \max_{y \in \overline{B_\varepsilon}(x)} \phi(s, y) + \min_{y \in \overline{B_\varepsilon}(x)} \phi(s, y) \right) ds - \phi(t, x) \\ & \geq \frac{\varepsilon^2}{2} \left( \int_{t-\varepsilon^2}^t \sum_{i,j=1}^{2n} X_i X_j \phi(t, x) \cdot \frac{(x_i^{\varepsilon,s} - x_i)}{\varepsilon} \cdot \frac{(x_j^{\varepsilon,s} - x_j)}{\varepsilon} ds - \phi_t(t, x) \right) + o(\varepsilon^2). \end{aligned} \quad (4.14)$$

This inequality holds for any function  $\phi \in C_H^2(\Omega_T)$ .

In the following, we prove the result via a dichotomy.

Because  $\phi \in C_H^2(\Omega)$ , if  $\nabla_0 \phi(t, x) \neq 0$ , so  $\nabla_0 \phi(s, x) \neq 0$  for  $t - \varepsilon^2 \leq s \leq t$  and for small enough  $\varepsilon > 0$ , and thus  $x^{\varepsilon,s} \in \partial B_\varepsilon(x)$  for small  $\varepsilon$ . By Lemma 4.1, we have

$$\lim_{\varepsilon \rightarrow 0} \frac{x_i^{\varepsilon,s} - x_i}{\varepsilon} = -\frac{X_i \phi}{|\nabla_0 \phi|}(t, x) \quad \text{for } i = 1, \dots, 2n. \quad (4.15)$$

Hence, we get the limit

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \int_{t-\varepsilon^2}^t \sum_{i,j=1}^{2n} X_i X_j \phi(t, x) \cdot \frac{(x_i^{\varepsilon,s} - x_i)}{\varepsilon} \cdot \frac{(x_j^{\varepsilon,s} - x_j)}{\varepsilon} ds \\ & = \sum_{i,j=1}^{2n} X_i X_j \phi(t, x) \cdot \frac{X_i \phi}{|\nabla_0 \phi|}(t, x) \cdot \frac{X_j \phi}{|\nabla_0 \phi|}(t, x) = \Delta_H^\infty \phi(t, x). \end{aligned} \quad (4.16)$$

We first prove that if the asymptotic mean value formula (1.17) holds for  $u$  in viscosity sense, then  $u$  satisfies the definition of viscosity solutions to (1.14) whenever  $\nabla_0 \phi \neq 0$ . Let  $\phi \in C_H^2(\Omega_T)$  be a test function such that  $u - \phi$  has a strict minimum at  $(t^0, x^0)$  and  $\nabla_0 \phi(t^0, x^0) \neq 0$ , we have

$$0 \geq -\phi(t^0, x^0) + \frac{1}{2} \int_{t^0-\varepsilon^2}^{t^0} \left( \max_{y \in \overline{B_\varepsilon}(x^0)} \phi(s, y) + \min_{y \in \overline{B_\varepsilon}(x^0)} \phi(s, y) \right) ds + o(\varepsilon^2). \quad (4.17)$$

By (4.14), (4.16) and (4.17), we have

$$o(\varepsilon^2) \geq \frac{\varepsilon^2}{2} (\Delta_H^\infty \phi(t^0, x^0) - \phi_t(t^0, x^0)) + o(\varepsilon^2). \quad (4.18)$$

Dividing by  $\varepsilon^2$  and passing to a limit, we get

$$\phi_t(t^0, x^0) \geq \Delta_H^\infty \phi(t^0, x^0). \quad (4.19)$$

That is  $u$  is a viscosity super-solution of (1.14).

To prove that  $u$  is a viscosity sub-solution, we first derive a reverse inequality to (4.14) by considering the maximum point of  $\phi$ , and then we choose a test function  $\phi$  that touches  $u$  from above.

To prove the reverse implication, assume that  $u$  is a viscosity super-solution of (1.14). Let  $\phi \in C_H^2(\Omega_T)$  be a test function such that  $u - \phi$  has a strict minimum at  $(t^0, x^0)$  and  $\nabla_0 \phi(t^0, x^0) \neq 0$ , we have

$$\Delta_H^\infty \phi(t^0, x^0) - \phi_t(t^0, x^0) \leq 0. \quad (4.20)$$

Dividing (4.14) by  $\varepsilon^2$ , using (4.16) and (4.20), we get

$$\limsup_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \left( -\phi(t^0, x^0) + \frac{1}{2} \int_{t^0 - \varepsilon^2}^{t^0} \left( \max_{y \in \overline{B}_\varepsilon(x^0)} \phi(s, y) + \min_{y \in \overline{B}_\varepsilon(x^0)} \phi(s, y) \right) ds \right) \leq 0. \quad (4.21)$$

That is

$$\phi(t^0, x^0) \geq \int_{t^0 - \varepsilon^2}^{t^0} \left( \max_{y \in \overline{B}_\varepsilon(x^0)} \phi(s, y) + \min_{y \in \overline{B}_\varepsilon(x^0)} \phi(s, y) \right) ds + o(\varepsilon^2). \quad (4.22)$$

Finally, let  $\phi \in C_H^2(\Omega_T)$  be a test function such that  $u - \phi$  has a strict minimum at  $(t^0, x^0)$  and  $\nabla_0 \phi(t^0, x^0) = 0$ . With the help of Theorem 1.3, we also assume that  $(X^2 \phi)^*(t^0, x^0) = 0$ , and thus the Taylor expansion (4.2) implies

$$\phi(s, y) - \phi(t^0, x^0) = \phi_t(t^0, x^0)(s - t^0) + T\phi(t^0, x^0) \left( y_{2n+1} - x_{2n+1}^0 + 2 \sum_{i=1}^n (y_{n+i} x_i - y_i x_{n+i}) \right) + o(\varepsilon^2).$$

That is

$$\begin{aligned} & \frac{1}{2} \left( \max_{y \in \overline{B}_\varepsilon(x^0)} (\phi(s, y) - \phi(t^0, x^0)) + \min_{y \in \overline{B}_\varepsilon(x^0)} (\phi(s, y) - \phi(t^0, x^0)) \right) \\ &= \phi_t(t^0, x^0)(s - t^0) + T\phi(t^0, x^0) \left\{ \max_{y \in \overline{B}_\varepsilon(x^0)} \left( y_{2n+1} - x_{2n+1}^0 + 2 \sum_{i=1}^n (y_{n+i} x_i^0 - y_i x_{n+i}^0) \right) \right. \\ & \quad \left. + \min_{y \in \overline{B}_\varepsilon(x^0)} \left( y_{2n+1} - x_{2n+1}^0 + 2 \sum_{i=1}^n (y_{n+i} x_i^0 - y_i x_{n+i}^0) \right) \right\} + o(\varepsilon^2). \end{aligned}$$

We claim

$$\max_{y \in \overline{B}_\varepsilon(x^0)} \left( y_{2n+1} - x_{2n+1}^0 + 2 \sum_{i=1}^n (y_{n+i} x_i^0 - y_i x_{n+i}^0) \right) + \min_{y \in \overline{B}_\varepsilon(x^0)} \left( y_{2n+1} - x_{2n+1}^0 + 2 \sum_{i=1}^n (y_{n+i} x_i^0 - y_i x_{n+i}^0) \right) = 0. \quad (4.23)$$

Indeed, if  $y \in \overline{B}_\varepsilon(x^0)$ , then

$$\begin{aligned} & \left( y_{2n+1} - x_{2n+1}^0 + 2 \sum_{i=1}^n (y_{n+i} x_i^0 - y_i x_{n+i}^0) \right)^2 \\ & \leq \left( \sum_{i=1}^{2n} (y_i - x_i^0)^2 \right) + \left( y_{2n+1} - x_{2n+1}^0 + 2 \sum_{i=1}^n (y_{n+i} x_i^0 - y_i x_{n+i}^0) \right)^2 \leq \varepsilon^4, \end{aligned}$$

thus

$$-\varepsilon^2 \leq y_{2n+1} - x_{2n+1}^0 + 2 \sum_{i=1}^n (y_{n+i} x_i^0 - y_i x_{n+i}^0) \leq \varepsilon^2.$$

Moreover, let

$$y_{\max} = (x_1^0, \dots, x_{2n}^0, x_{2n+1}^0 + \varepsilon^2) \in \overline{B}_\varepsilon(x^0),$$

and

$$y_{\min} = (x_1^0, \dots, x_{2n}^0, x_{2n+1}^0 - \varepsilon^2) \in \overline{B}_\varepsilon(x^0).$$

The maximum and minimum value can achieve at  $y_{\max}$  and  $y_{\min}$ , respectively, i.e.

$$\max_{y \in \overline{B}_\varepsilon(x^0)} \left( y_{2n+1} - x_{2n+1}^0 + 2 \sum_{i=1}^n (y_{n+i} x_i^0 - y_i x_{n+i}^0) \right) = \varepsilon^2,$$

and

$$\min_{y \in \overline{B}_\varepsilon(x^0)} \left( y_{2n+1} - x_{2n+1}^0 + 2 \sum_{i=1}^n (y_{n+i} x_i^0 - y_i x_{n+i}^0) \right) = -\varepsilon^2.$$

This ends the proof of the claim (4.23). Therefore

$$\frac{1}{2} \left( \max_{y \in \overline{B}_\varepsilon(x^0)} (\phi(s, y) - \phi(t^0, x^0)) + \min_{y \in \overline{B}_\varepsilon(x^0)} (\phi(s, y) - \phi(t^0, x^0)) \right) = \phi_t(t^0, x^0)(s - t^0) + o(\varepsilon^2). \quad (4.24)$$

Suppose that the asymptotic mean value formula (4.10) holds at  $(t^0, x^0)$ , we get

$$\phi(t^0, x^0) \geq \frac{1}{2} \int_{t^0 - \varepsilon^2}^{t^0} \left( \max_{y \in \overline{B}_\varepsilon(x^0)} \phi(s, y) + \min_{y \in \overline{B}_\varepsilon(x^0)} \phi(s, y) \right) ds + o(\varepsilon^2).$$

Hence, by (4.24), we have

$$\begin{aligned} 0 &\geq \frac{1}{2} \int_{t^0 - \varepsilon^2}^{t^0} \left( \max_{y \in \overline{B}_\varepsilon(x^0)} (\phi(s, y) - \phi(t^0, x^0)) + \min_{y \in \overline{B}_\varepsilon(x^0)} (\phi(s, y) - \phi(t^0, x^0)) \right) ds + o(\varepsilon^2) \\ &= \int_{t^0 - \varepsilon^2}^{t^0} \phi_t(t^0, x^0)(s - t^0) ds + o(\varepsilon^2) \\ &= -\frac{\varepsilon^2}{2} \phi_t(t^0, x^0) + o(\varepsilon^2). \end{aligned} \quad (4.25)$$

Dividing (4.25) by  $\varepsilon^2$  and passing to a limit, we obtain

$$\phi_t(t^0, x^0) \geq 0.$$

Thus, Theorem 1.3 shows  $u$  is a viscosity super-solution of (1.14).

Suppose that  $u$  is a viscosity super-solution of (1.14). Let  $\phi \in \Omega_T$  be a test function such that  $u - \phi$  has a strict minimum at  $(t^0, x^0)$ ,  $\nabla_0 \phi(t^0, x^0) = 0$  and  $(X^2 \phi)^*(t^0, x^0) = 0$ , we have

$$\phi_t(t^0, x^0) \geq 0.$$

By (4.24), we get

$$\begin{aligned} &\frac{1}{2} \int_{t^0 - \varepsilon^2}^{t^0} \left( \max_{y \in \overline{B}_\varepsilon(x^0)} \phi(s, y) + \min_{y \in \overline{B}_\varepsilon(x^0)} \phi(s, y) \right) ds - \phi(t^0, x^0) \\ &= \int_{t^0 - \varepsilon^2}^{t^0} \phi_t(t^0, x^0)(s - t^0) ds + o(\varepsilon^2) \\ &= -\frac{\varepsilon^2}{2} \phi_t(t^0, x^0) + o(\varepsilon^2) \leq o(\varepsilon^2). \end{aligned}$$

Thus, dividing the above equality by  $\varepsilon^2$  and passing to a limit, we have

$$\phi(t^0, x^0) \geq \int_{t^0 - \varepsilon^2}^{t^0} \left( \max_{y \in \overline{B}_\varepsilon(x^0)} \phi(s, y) + \min_{y \in \overline{B}_\varepsilon(x^0)} \phi(s, y) \right) ds + o(\varepsilon^2).$$

Therefore, the proof of the theorem is complete.  $\square$

Combining the case  $p = 2$  with the case  $p = \infty$ , we prove the general case  $1 < p < \infty$ .

*Proof of Theorem 1.4* Assume that  $p \geq 2$  so that  $\alpha \geq 0$ . Multiplying (4.7) by  $\beta$ , (4.14) by  $\alpha$ , and adding, we get

$$\begin{aligned} &\frac{\alpha}{2} \int_{t - \varepsilon^2}^t \left( \max_{y \in \overline{B}_\varepsilon(x)} \phi(s, y) + \min_{y \in \overline{B}_\varepsilon(x)} \phi(s, y) \right) ds + \beta \int_{t - \varepsilon^2}^t \int_{B_\varepsilon(x)} \psi(x^{-1} \circ y) \phi(s, y) dy ds - \phi(x, t) \\ &\geq \frac{\alpha}{2} \varepsilon^2 \left( \int_{t - \varepsilon^2}^t \sum_{i,j=1}^{2n} X_i X_j \phi(x, t) \cdot \frac{(x_i^{\varepsilon,s} - x_i)}{\varepsilon} \cdot \frac{(x_j^{\varepsilon,s} - x_j)}{\varepsilon} ds - \phi_t(t, x) \right) + \frac{\beta}{2} \varepsilon^2 (M(n) \Delta_H \phi(t, x) - \phi_t(t, x)) + o(\varepsilon^2) \\ &= \frac{\beta}{2} \varepsilon^2 \left( \frac{\alpha}{\beta} \int_{t - \varepsilon^2}^t \sum_{i,j=1}^{2n} X_i X_j \phi(x, t) \cdot \frac{(x_i^{\varepsilon,s} - x_i)}{\varepsilon} \cdot \frac{(x_j^{\varepsilon,s} - x_j)}{\varepsilon} ds + M(n) \Delta_H \phi(t, x) - \left( \frac{\alpha}{\beta} + 1 \right) \phi_t(t, x) \right) + o(\varepsilon^2). \end{aligned}$$

Thanks to

$$\begin{cases} \beta M(n)(p-2) = \alpha, \\ \alpha + \beta = 1, \end{cases}$$

we have

$$\begin{aligned} & \frac{\alpha}{2} \int_{t-\varepsilon^2}^t (\max_{y \in \bar{B}_\varepsilon(x)} \phi(s, y) + \min_{y \in \bar{B}_\varepsilon(x)} \phi(s, y)) ds + \beta \int_{t-\varepsilon^2}^t \int_{B_\varepsilon(x)} \psi(x^{-1} \circ y) \phi(s, y) dy ds - \phi(x, t) \\ & \geq \frac{\beta}{2} \varepsilon^2 (M(n)(p-2) \int_{t-\varepsilon^2}^t \sum_{i,j=1}^{2n} X_i X_j \phi(x, t) \cdot \frac{(x_i^{\varepsilon, s} - x_i)}{\varepsilon} \cdot \frac{(x_j^{\varepsilon, s} - x_j)}{\varepsilon} ds \\ & \quad + M(n) \Delta_H \phi(t, x) - (M(n)(p-2) + 1) \phi_t(x, t)) + o(\varepsilon^2). \end{aligned} \quad (4.26)$$

The rest proof follows that of Theorem 1.2. Furthermore, by considering the maximum point instead of the minimum point, we can get a reverse inequality to (4.26).

If  $1 < p < 2$ , it follows that  $\alpha < 0$ , and the inequality (4.26) is reversed. On the other hand, so is the reverse inequality that can be obtained by considering the maximum point instead of the minimum point. The argument then continues to work in the same way as before.  $\square$

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